

A problem of I. Raşa on Bernstein polynomials and convex functions

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Abstract.

We present an elementary proof of a conjecture by I. Raşa which is an inequality involving Bernstein basis polynomials and convex functions. It was affirmed in positive very recently by the use of stochastic convex orderings. Moreover, we derive the corresponding results for Mirakyan-Favard-Szász operators and Baskakov operators.

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1 Introduction

For $n = 0, 1, 2, \dots$ and $\nu = 0, 1, \dots, n$, let

$$p_{n,\nu}(x) = \binom{n}{\nu} x^\nu (1-x)^{n-\nu}$$

denote the Bernstein basis polynomials. For $\nu > n$ we define $p_{n,\nu}(x) = 0$. Very recently, J. Mrowiec, T. Rajba and S. Wąsowicz [2] proved the following theorem.

Theorem 1 ([2]) *Let $n \in \mathbb{N}$. If $f \in C[0, 1]$ is a convex function, then*

$$\sum_{i=0}^n \sum_{j=0}^n [p_{n,i}(x) p_{n,j}(x) + p_{n,i}(y) p_{n,j}(y) - 2p_{n,i}(x) p_{n,j}(y)] f\left(\frac{i+j}{2n}\right) \geq 0, \quad (1)$$

for all $x, y \in [0, 1]$.

This inequality involving Bernstein basis polynomials and convex functions was stated as an open problem 25 years ago by I. Raşa. J. Mrowiec, T. Rajba and S. Wąsowicz [2] affirmed the conjecture in positive. Their proof makes heavy use of probability theory. As a tool they applied stochastic convex orderings (which they proved for binomial distributions) as well as the so-called concentration inequality.

The purpose of this note is a short elementary proof of the above theorem. It even doesn't take advantage of the Levin-Stečkin theorem [1] (see also [3], Theorem 4.2.7) which gives necessary and sufficient conditions on F for the non-negativity of $\int_a^b f(x) dF(x)$ if f is a convex function on $[a, b]$.

In the last two sections we prove the corresponding results for Mirakyan-Favard-Szász operators and Baskakov operators.

2 An elementary proof of Theorem 1

We start with a few auxiliary results.

Lemma 2 For $n, k \in \mathbb{N}$,

$$\sum_{i=0}^k p_{n,i}(x) p_{n,k-i}(y) = \frac{1}{k!} \left[\left(\frac{\partial}{\partial z} \right)^k (1+xz)^n (1+yz)^n \right]_{|z=-1}.$$

Note that the formula is valid also if $k > n$.

Proof. We have

$$\begin{aligned} & \sum_{i=0}^k p_{n,i}(x) p_{n,k-i}(y) \\ &= \sum_{i=0}^k \binom{n}{i} x^i (1-x)^{n-i} \binom{n}{k-i} y^{k-i} (1-y)^{n-(k-i)} \\ &= \frac{1}{k!} \left\{ \sum_{i=0}^k \binom{k}{i} \left[\left(\frac{\partial}{\partial z} \right)^i (1+xz)^n \right] \left[\left(\frac{\partial}{\partial z} \right)^{k-i} (1+yz)^n \right] \right\}_{|z=-1} \end{aligned}$$

and the lemma follows by an application of the Leibniz rule for the differentiation of products of functions. ■

The next result is a representation of the left-hand side of Eq. (1).

Lemma 3

$$\begin{aligned} & \sum_{i=0}^n \sum_{j=0}^n [p_{n,i}(x) p_{n,j}(x) + p_{n,i}(y) p_{n,j}(y) - 2p_{n,i}(x) p_{n,j}(y)] f\left(\frac{i+j}{2n}\right) \\ &= \sum_{k=0}^{2n} f\left(\frac{k}{2n}\right) \frac{1}{k!} \left[\left(\frac{\partial}{\partial z} \right)^k [(1+xz)^n - (1+yz)^n]^2 \right]_{|z=-1}. \end{aligned}$$

Proof. It is a direct consequence of the preceding lemma that

$$\begin{aligned} & \sum_{i=0}^n \sum_{j=0}^n [p_{n,i}(x)p_{n,j}(x) + p_{n,i}(y)p_{n,j}(y) - 2p_{n,i}(x)p_{n,j}(y)] f\left(\frac{i+j}{2n}\right) \\ &= \sum_{k=0}^{2n} f\left(\frac{k}{2n}\right) \frac{1}{k!} \left(\frac{\partial}{\partial z}\right)^k [(1+xz)^{2n} + (1+yz)^{2n} - 2(1+xz)^n(1+yz)^n] \Big|_{z=-1} \end{aligned}$$

and the lemma follows, by the binomial formula. ■

For fixed $n \in \mathbb{N}$ and $x, y \in [0, 1]$, we define

$$g(z) \equiv g_n(z; x, y) = \left(\frac{(1+xz)^n - (1+yz)^n}{z} \right)^2.$$

Note that g is a polynomial in z of degree at most $2n - 2$.

The next proposition is the key result.

Proposition 4 *Let $(a_k)_{k=0}^{2n}$ be a real sequence and fix $x, y \in [0, 1]$. Then,*

$$\sum_{i=0}^n \sum_{j=0}^n [p_{n,i}(x)p_{n,j}(x) + p_{n,i}(y)p_{n,j}(y) - 2p_{n,i}(x)p_{n,j}(y)] \cdot a_{i+j} = \sum_{k=0}^{2n-2} (\Delta^2 a_k) \frac{1}{k!} g^{(k)}(-1) \quad (2)$$

and $g^{(k)}(-1) \geq 0$, for $k = 0, 1, \dots, 2n - 2$.

Here Δ denotes the forward difference $\Delta a_k := a_{k+1} - a_k$ such that $\Delta^2 a_k = a_{k+2} - 2a_{k+1} + a_k$.

Because g is a polynomial in z of degree at most $2n - 2$, it is obvious that $g^{(2n-1)}(-1) = g^{(2n)}(-1) = 0$.

Proof. Observe that

$$(z^2 g(z))^{(k)} = z^2 g^{(k)}(z) + \binom{k}{1} \cdot 2z g^{(k-1)}(z) + \binom{k}{2} \cdot 2g^{(k-2)}(z).$$

By Lemma 3, we have

$$\begin{aligned} & \sum_{i=0}^n \sum_{j=0}^n [p_{n,i}(x)p_{n,j}(x) + p_{n,i}(y)p_{n,j}(y) - 2p_{n,i}(x)p_{n,j}(y)] \cdot a_{i+j} \\ &= \sum_{k=0}^{2n} a_k \frac{1}{k!} \left(\frac{\partial}{\partial z}\right)^k (z^2 g(z)) \Big|_{z=-1} \\ &= \sum_{k=0}^{2n-2} a_k \frac{1}{k!} g^{(k)}(-1) - 2 \sum_{k=1}^{2n-1} a_k \frac{1}{(k-1)!} g^{(k-1)}(-1) + \sum_{k=2}^{2n} a_k \frac{1}{(k-2)!} g^{(k-2)}(-1) \\ &= \sum_{k=0}^{2n-2} (a_k - 2a_{k+1} + a_{k+2}) \frac{1}{k!} g^{(k)}(-1) \end{aligned}$$

which proves Eq. (2). Noting that

$$\begin{aligned} g(z) &= (x-y)^2 \left(\frac{(1+xz)^n - (1+yz)^n}{(1+xz) - (1+yz)} \right)^2 \\ &= (x-y)^2 \left(\sum_{k=0}^{n-1} (1+xz)^k (1+yz)^{n-1-k} \right)^2 \end{aligned}$$

it is immediate that $g^{(k)}(-1) \geq 0$, for $k = 0, 1, \dots, 2n-2$, if $x, y \in [0, 1]$. ■

Using the elementary formula

$$\left(\frac{b^n - a^n}{b - a} \right)^2 = \sum_{j=0}^{n-1} a^j b^{2n-2-j} \min \{j, 2n-2-j\}$$

we obtain more precisely

$$g(z) = (x-y)^2 \sum_{j=0}^{2n-2} (1+xz)^j (1+yz)^{2n-2-j} \min \{j, 2n-2-j\}.$$

Hence,

$$\begin{aligned} g^{(k)}(-1) &= k! (x-y)^2 \sum_{j=0}^{2n-2} \min \{j, 2n-2-j\} \\ &\quad \times \sum_{i=0}^k \binom{j}{i} x^i (1-x)^{j-i} \binom{2n-2-j}{k-i} y^{k-i} (1-y)^{2n-2-j-(k-i)} \\ &= k! (x-y)^2 \sum_{j=0}^{2n-2} \min \{j, 2n-2-j\} \sum_{i=0}^k p_{j,i}(x) p_{2n-2-j,k-i}(y). \end{aligned}$$

Proof of Theorem 1. For $k = 0, 1, \dots, 2n-2$, we put

$$a_k = f\left(\frac{k}{2n}\right).$$

If $f \in C[0, 1]$ is a convex function it follows that $\Delta^2 a_k \geq 0$, for $k = 0, 1, \dots, 2n-2$. Therefore, application of Proposition 4 proves Theorem 1. ■

3 Mirakyan-Favard-Szász operators

The Mirakyan-Favard-Szász S_n operators associate to each function f of (at most) exponential growth on $[0, \infty)$ the function

$$(S_n f)(x) = e^{-nx} \sum_{\nu=0}^{\infty} \frac{(nx)^\nu}{\nu!} f\left(\frac{\nu}{n}\right) \quad (x \in [0, \infty)).$$

If, for $n, \nu = 0, 1, 2, \dots$,

$$s_\nu(x) = e^{-x} \frac{x^\nu}{\nu!}$$

denote the corresponding basis functions, the operators can be written in the form

$$(S_n f)(x) = \sum_{\nu=0}^{\infty} s_\nu(nx) f\left(\frac{\nu}{n}\right) \quad (x \in [0, \infty)).$$

Theorem 5 *Let $n \in \mathbb{N}$. If $f \in C[0, \infty)$ is a convex function of (at most) exponential growth, then*

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} [s_i(x) s_j(x) + s_i(y) s_j(y) - 2s_i(x) s_j(y)] f(i+j) \geq 0,$$

for all $x, y \in [0, \infty)$.

Proof. It can easily be verified that

$$\sum_{i=0}^k s_i(x) s_{k-i}(y) = \frac{1}{k!} (x+y)^k e^{-(x+y)} = \frac{1}{k!} \left[\left(\frac{\partial}{\partial z} \right)^k e^{(x+y)z} \right]_{|z=-1}.$$

Hence,

$$\begin{aligned} & \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} [s_i(x) s_j(x) + s_i(y) s_j(y) - 2s_i(x) s_j(y)] f(i+j) \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} f(k) \left[\left(\frac{\partial}{\partial z} \right)^k (e^{2xz} + e^{2yz} - 2e^{(x+y)z}) \right]_{|z=-1} \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} f(k) \left[\left(\frac{\partial}{\partial z} \right)^k (e^{xz} - e^{yz})^2 \right]_{|z=-1}. \end{aligned}$$

Now we put, for fixed $x, y \geq 0$,

$$g(z) = \left(\frac{e^{xz} - e^{yz}}{z} \right)^2.$$

Observe that

$$g(z) = \int_x^y \int_x^y e^{(u+v)z} du dv = \sum_{\nu=0}^{\infty} \frac{(z+1)^\nu}{\nu!} \int_x^y \int_x^y (u+v)^\nu e^{-(u+v)} du dv$$

which implies that $g^{(k)}(-1) \geq 0$, for $k = 0, 1, \dots$, if $x, y \geq 0$. As in the Bernstein case we conclude that

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} [s_i(x) s_j(x) + s_i(y) s_j(y) - 2s_i(x) s_j(y)] f(i+j)$$

$$\begin{aligned}
&= \sum_{k=0}^{\infty} f(k) \frac{1}{k!} \left(\frac{\partial}{\partial z} \right)^k (z^2 g(z)) \Big|_{z=-1} \\
&= \sum_{k=0}^{\infty} \Delta^2 f(k) \cdot \frac{1}{k!} g^{(k)}(-1)
\end{aligned}$$

which completes the proof. ■

4 Baskakov operators

The Baskakov operators V_n associate to each function f of polynomial growth on $[0, \infty)$ the function

$$(V_n f)(x) = \sum_{\nu=0}^{\infty} b_{n,\nu}(x) f\left(\frac{\nu}{n}\right) \quad (x \in [0, \infty)),$$

where

$$b_{n,\nu}(x) = \binom{n+\nu-1}{\nu} \frac{x^\nu}{(1+x)^{n+\nu}}$$

denote the Baskakov basis functions. We have

$$\sum_{i=0}^k b_{n,i}(x) b_{n,k-i}(y) = \frac{1}{k!} \left[\left(\frac{\partial}{\partial z} \right)^k (1-xz)^{-n} (1-yz)^{-n} \right] \Big|_{z=-1}$$

and

$$\begin{aligned}
&\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} b_{n,i}(x) b_{n,j}(x) [b_{n,i}(x) b_{n,j}(x) + b_{n,i}(y) b_{n,j}(y) - 2b_{n,i}(x) b_{n,j}(y)] f(i+j) \\
&= \sum_{k=0}^{\infty} f(k) \frac{1}{k!} \left[\left(\frac{\partial}{\partial z} \right)^k ((1-xz)^{-n} - (1+yz)^{-n})^2 \right] \Big|_{z=-1}
\end{aligned}$$

In a similar manner as in the Bernstein case one can show the following theorem.

Theorem 6 *Let $n \in \mathbb{N}$. If $f \in C[0, \infty)$ is a convex function of polynomial growth, then*

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} [b_{n,i}(x) b_{n,j}(x) + b_{n,i}(y) b_{n,j}(y) - 2b_{n,i}(x) b_{n,j}(y)] f(i+j) \geq 0,$$

for all $x, y \in [0, \infty)$.

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